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$\boldsymbol{R(s)}$ and hadronic $\boldsymbol{\tau}$-Decays in Order $\boldsymbol{\alpha_s^4}$: technical aspects*

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We report on some technical aspects of our calculation of α_s^4 corrections to $R(s)$ and the semi-leptonic τ decay width [1,2,3]. We discuss the inner structure of the result as well as the issue of its correctness. We demonstrate recently appeared *independent evidence* **positively** testing *one* of two components of our full result.

1. Introduction

Three important physical observables, namely, the ratio $R(s) = \frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}$, the hadronic decay rate of the Z-boson and the semileptonic branching ratio of the τ-lepton are expressed through the vector and axial-vector current correlators (see, e. g. reviews [4,5]). Perturbative QCD provides reliable predictions for these correlators in the continuum, i.e. sufficiently above the respective quark threshold and the respective resonance region.

The $\mathcal{O}(\alpha_s^3)$ result for the massless vector correlator[3] has been known since many years [6,7].

Recently the calculation of the next, order α_s^4 contribution to the vector correlator has been performed [1,2,3]. The aim of the present work is to discuss some technical aspects of our calculations as well as to provide some new arguments in favour of their correctness.

Due to lack of space no phenomenological implications of [1,2] are discussed and the interested reader is referred to Refs. [1,8,9,10,11,12].

2. Generalities

Consider the two-point correlator of vector quark currents and the corresponding vacuum polarization function ($j_\mu^v = \overline{Q}\gamma_\mu Q$; Q is a quark field with mass m, all other $n_f - 1$ quarks are assumed to be massless)

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= i\int \mathrm{d}x e^{iqx}\langle 0|T[\ j_\mu^v(x) j_\nu^v(0)\ ]|0\rangle \\
&= (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(q^2). \qquad (1)
\end{aligned}
$$

The physical observable $R(s)$ is related to $\Pi(q^2)$ by

$$R(s) = 12\pi \Im\, \Pi(q^2 + i\epsilon). \qquad (2)$$

For future reference it is convenient to decompose $R(s)$ into the massless contribution and the one quadratic in the quark mass as follows ($a_s = \alpha_s(\mu^2)/\pi$):

$$
\begin{aligned}
R(s) &= 3\left\{ r_0^V + \frac{m^2}{s} r_2^V \right\} + \ldots \\
&= 3\left\{ \sum_{i\geq 0} a_s^i \left(r_0^{V,i} + \frac{m^2}{s} r_2^{V,i} \right) \right\} + \ldots .
\end{aligned}
$$

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[3] Note that in the massless limit vector and axial-vector correlators are equal *provided* one considers only non-singlet contributions and ignores so-called singlet ones. The latter are absent in the tau-lepton case and numerically small for the Z-boson decay rate. In the present work we will discuss non-singlet contributions only.

The corresponding representation for the polarization function reads ($Q^2 \equiv -q^2$)

$$\Pi = \Pi_0(L, a_s) + \frac{m^2}{Q^2}\Pi_2(L, a_s) + \mathcal{O}(1/Q^4). \qquad (3)$$

Note that both function on the rhs of (3) depend on only a_s and $L = \ln\frac{\mu^2}{Q^2}$ and could be conveniently decomposed as follows ($n = 0, 2$)

$$\Pi_n = \sum_{i\geq 0} \Pi_i^n \, a_s^i, \quad \Pi_i^n = \sum_{0\leq j\leq i+1} \Pi_{i,j}^n \, L^j. \qquad (4)$$

The terms in (4) without L-dependence do not contribute to $R(s)$.

For the calculation of $r_0^{V,4}$ the divergent parts of five-loop and the finite parts of the four-loop diagrams are needed [13]. The organization of the calculation is best based on using of the evolution equation for Π (see, e. g. [14])

$$\frac{\partial}{\partial L}\Pi_0 = \gamma^{VV}(a_s) - \left(\beta(a_s) a_s \frac{\partial}{\partial a_s}\right)\Pi_0, \qquad (5)$$

where $\gamma^{VV} = \sum_{i\geq 0} \gamma_i^{VV} a_s^i$ is the (subtractive) anomalous dimension of the correlator (1) and $\beta = -\sum_{i\geq 0} \beta_i \, a_s^{(i+1)}$ is the QCD β-function.

To evaluate the L-dependent pieces of the polarization function $\Pi_0^0 \dots \Pi_4^0$ in terms of $\gamma_0^{VV} \dots \gamma_4^{VV}$ and $\Pi_0^0 \dots \Pi_3^0$ the evolution eq. (5) can be solved perturbatively:

$$\Pi_0^0 = \gamma_0^{VV} \, L + \Pi_{0,0}^0, \quad \Pi_1^0 = \gamma_1^{VV} \, L + \Pi_{1,0}^0, \qquad (6)$$

$$\Pi_2^0 = \beta_0 \, \gamma_1^{VV} \, \tfrac{L^2}{2} + L \, \left(\gamma_2^{VV} + \beta_0 \, \Pi_{1,0}^0\right) + \Pi_{2,0}^0, \quad (7)$$

$$\begin{aligned}\Pi_3^0 &= \beta_0^2 \, \gamma_1^{VV} \, \tfrac{L^3}{3} \\ &+ \tfrac{L^2}{2} \, \left(\beta_1 \, \gamma_1^{VV} + 2\,\beta_0 \, \gamma_2^{VV} + 2\,\beta_0^2 \, \Pi_{1,0}^0\right) \\ &\quad + L \, \left(\gamma_3^{VV} + \beta_1 \, \Pi_{1,0}^0 + 2\,\beta_0 \, \Pi_{2,0}^0\right) + \Pi_{3,0}^0,\end{aligned} \qquad (8)$$

$$\begin{aligned}\Pi_4^0 &= \beta_0^3 \, \gamma_1^{VV} \, \tfrac{L^4}{4} + L^3 \, \left(\tfrac{5}{6}\,\beta_0 \, \beta_1 \, \gamma_1^{VV} + \beta_0^2 \, \gamma_2^{VV}\right. \\ &\left.+ \beta_0^3 \, \Pi_{1,0}^0\right) + L^2 \left(\tfrac{1}{2}\beta_2 \, \gamma_1^{VV} + \beta_1 \, \gamma_2^{VV} + \tfrac{3}{2}\,\beta_0 \, \gamma_3^{VV}\right. \\ &\left.+ \tfrac{5}{2}\,\beta_0 \, \beta_1 \, \Pi_{1,0}^0 + 3\,\beta_0^2 \, \Pi_{2,0}^0\right) + L \left(\gamma_4^{VV} + \beta_2 \, \Pi_{1,0}^0\right. \\ &\left.+ 2\,\beta_1 \, \Pi_{2,0}^0 + 3\,\beta_0 \, \Pi_{3,0}^0\right) + \Pi_{4,0}^0.\end{aligned} \qquad (9)$$

The evolution equation for Π_2 describing the m^2-corrections looks similar to (5), namely [3,15],

$$\frac{\partial}{\partial L}\Pi_2 = -\left(2\,\gamma_m(a_s) + \beta(a_s) a_s \frac{\partial}{\partial a_s}\right)\Pi_2, \qquad (10)$$

where $\gamma_m = -\sum_{i\geq 0} \gamma_m^{(i+1)} a_s^i$ is the quark mass anomalous dimension.

3. Results

We refer the reader to [1] for a discussion of various theoretical tools used to compute Π_0 to order a_s^3 and γ^{VV} to a_s^4. We only want to mention here the indispensable role of the parallel version [16] of FORM [17] and the availability of large computing resources. The results for γ^{VV} and Π are given in the next two subsections.

3.1. Five loop anomalous dimension

$$(4\pi)^2 \, \gamma_0^{VV} = (4\pi)^2 \, \gamma_1^{VV} = 4, \qquad (11)$$

$$(4\pi)^2 \, \gamma_2^{VV} = -\frac{11}{18} n_f + \frac{125}{12}, \qquad (12)$$

$$\begin{aligned}(4\pi)^2 \, \gamma_3^{VV} &= -\frac{77}{972} \, n_f^2 + n_f \left[-\frac{707}{216} - \frac{110}{27}\,\zeta_3\right] \\ &+ \frac{10487}{432} + \frac{110}{9}\,\zeta_3,\end{aligned} \qquad (13)$$

$$\begin{aligned}(4\pi)^2 \, \gamma_4^{VV} &= n_f^3 \left[\frac{107}{15552} + \frac{1}{108}\,\zeta_3\right] \\ &+ n_f^2 \left[-\frac{4729}{31104} + \frac{3163}{1296}\,\zeta_3 - \frac{55}{72}\,\zeta_4\right] \\ &+ n_f \left[-\frac{11785}{648} - \frac{58625}{864}\,\zeta_3 + \frac{715}{48}\,\zeta_4 + \frac{13325}{432}\,\zeta_5\right] \\ &+ \frac{2665349}{41472} + \frac{182335}{864}\,\zeta_3 - \frac{605}{16}\,\zeta_4 - \frac{31375}{288}\,\zeta_5.\end{aligned} \qquad (14)$$

3.2. $\mathcal{O}(\alpha_s^3)$ polarization operator

$$(4\pi)^2 \, \Pi_{0,0}^0 = \frac{20}{3}, \quad (4\pi)^2 \, \Pi_{1,0}^0 = \frac{55}{3} - 16\,\zeta_3, \qquad (15)$$

$$\begin{aligned}(4\pi)^2 \, \Pi_{2,0}^0 &= n_f \left[-\frac{3701}{324} + \frac{76}{9}\,\zeta_3\right] \\ &+ \frac{41927}{216} - \frac{1658}{9}\,\zeta_3 + \frac{100}{3}\,\zeta_5,\end{aligned} \qquad (16)$$

$$(4\pi)^2\,\Pi^0_{3,0} = n_f^2\left[\frac{196513}{23328} - \frac{809}{162}\zeta_3 - \frac{20}{9}\zeta_5\right]$$
$$+ \quad n_f\left[-\frac{1863319}{5184} + \frac{174421}{648}\zeta_3 - \frac{20}{3}\zeta_3^2 - \frac{55}{36}\zeta_4 + \frac{1090}{27}\zeta_5\right]$$
$$+ \quad \frac{31431599}{10368} - \frac{624799}{216}\zeta_3 + 330\,\zeta_3^2 + \frac{55}{12}\zeta_4 + \frac{1745}{24}\zeta_5 - \frac{665}{9}\zeta_7\,. \qquad (17)$$

The results for the analytical calculation of $\Pi^2_{0,0}\ldots\Pi^2_{3,0}$ have been reported in [3] while the four-loop quark anomalous dimension is known from [18,19] and three-loop QCD β-function from [20,21]. Numerically we find for the polarization operator

$$\Pi^0_0 = 0.0422172 + 0.0253303\,L, \qquad (18)$$
$$\Pi^0_1 = -0.00569664 + 0.0253303\,L, \qquad (19)$$
$$\Pi^0_2 = 0.0457538 - 0.0080559\,n_f, \qquad (20)$$
$$+(0.0502986 - 0.00292047\,n_f)\,L,$$
$$+(0.0348292 - 0.00211086\,n_f)\,L^2,$$
$$\Pi^0_3 = \qquad (21)$$
$$+0.23570 - 0.033603\,n_f + 0.0007394\,n_f^2$$
$$+(0.462093 - 0.10679\,n_f + 0.0021836\,n_f^2)\,L$$
$$+(0.219061 - 0.02644\,n_f + 0.00048674\,n_f^2)\,L^2$$
$$+(0.063854 - 0.0077398\,n_f + 0.0002345\,n_f^2)\,L^3$$

and

$$\Pi^2_0 = -0.151982, \qquad (22)$$
$$\Pi^2_1 = -0.405285 - 0.303964L, \qquad (23)$$
$$\Pi^2_2 = -4.27066 + 0.200532\,n_f, \qquad (24)$$
$$+(-3.20428 + 0.109765\,n_f)\,L,$$
$$+(-0.721913 + 0.0253303\,n_f)\,L^2,$$
$$\Pi^2_3 = \qquad (25)$$
$$-53.0381 + 5.21239\,n_f - 0.0740141\,n_f^2$$
$$+(-43.9568 + 4.05526\,n_f - 0.0586349\,n_f^2)\,L$$
$$+(-14.2641 + 1.1082\,n_f - 0.0182941\,n_f^2)\,L^2$$
$$+(-1.80478 + 0.143538\,n_f - 0.00281448\,n_f^2)\,L^3.$$

Note that at eqs. (11-13,15-16) and (18-20,22-24) as well as L-dependent pieces of of eqs. (21,25) are, in fact, known since long [6,7,22,15].

ℓ	1	2	3	4
	-	ζ_3	ζ_3,ζ_4,ζ_5	$\zeta_3,\zeta_4,\zeta_5,\zeta_3^2,\zeta_6,\zeta_7$

Table 1
Possible irrational structures which are allowed to appear in ℓ-loop massless propagators.

ℓ	1,2	3	4	5
	-	ζ_3	ζ_3,ζ_4,ζ_5	$\zeta_3,\zeta_4,\zeta_5,\zeta_3^2,\zeta_6,\zeta_7$

Table 2
Possible irrational structures which are allowed to appear in ℓ-loop anomalous dimensions and β-functions.

4. Final results for $R(s)$

Our final results for $r_0^{V,i}$ and $r_2^{V,i}$ are easily obtained from results listed in the previous two subsections. Explicit expressions can be found in [1,3].

It is of interest to discuss the structure of trancendentalities appearing in eqs. (11-17). On general grounds one could expect that the variety of ζ-constants entering into $\overline{\rm MS}$-renormalized (euclidian) massless propagators[4] should depend on the loop order according to Table 1. Table 2 provides the same information about possible irrational numbers which could show up in anomalous dimensions. Table 2 comes directly from Table 1 by noting that any $\ell+1$-loop anomalous dimension can be obtained from properly chosen ℓ-loop massless propagators [23].

An examination of eqs. (11-17) immediately reveals that the real pattern of trancendentalities is significantly more limited than what is allowed by Tables 1 and 2. Indeed, the four-loop anomalous dimension γ_3^{VV} contains no ζ_4 and no ζ_5 while the three-loop polarization operator contains ζ_5 but does not comprises ζ_4. The fact of absence

[4] It is understood that $\mathcal{O}(\epsilon^{(5-\ell)}$ terms in an ℓ-loop massless propagator could contribute only to six-loop anomalous dimension and, thus, are not constrained by Table 1.

of ζ_4 in $\mathcal{O}(\alpha_s^3)$ contribution to the Adler function is well-known and well-understood [6,24]. Why γ_3^{VV} is free from ζ_5 is still unclear (at least for us).

Let us move up one loop. The situation is getting even more intriguing: the five-loop anomalous dimension γ_4^{VV} does contain ζ_4 but still does not include ζ_3^2, ζ_6 and ζ_7. The four-loop polarization operator contains ζ_4 but is free from ζ_6. Even more, after we combine γ^{VV} and Π^0 to produce the Adler function, the resulting coefficient in front of ζ_4 happens to be zero in a non-trivial way! Indeed, the contribution proportional to ζ_4 from $\Pi^0_{3,0}$ reads

$$3\beta_0\left(\tfrac{55}{12} - \tfrac{55}{36}\, n_f\right) = -\left(-\tfrac{605}{16} + \tfrac{715}{48}\, n_f - \tfrac{55}{72}\, n_f^2\right)$$

and is *exactly* opposite in sign to the corresponding piece in (14)!

Unfortunately, we are not aware about existence of any ratio behind these remarkable observations.

5. How reliable are our results?

The history of multiloop calculations teaches us to be cautious. For instance, approximately twenty years ago a severely wrong result for the $\mathcal{O}(\alpha_s^3)$ coefficient in $R(s)$ was published [25] and corrected only three years later [6,7].

Now one of these authors (rightfully!) rises an important issue of the correctness of the results [1,2] and emphasizes the necessity of performing their independent test [26,27].

We completely agree with this argumentation. Unfortunately, at the moment, we are not aware of any independent team which is going or, at least, able to check our results in full.

However, as described below, the results of a recent calculation [28] allow at least for a (partial) test of [1,2,3].

5.1. A test of the polarization operator

In Ref. [28] a large amount of information about the *massive four-loop polarization* function was collected (its threshold behavior [29,30,31], as well as low-energy moments [32,33,34] and high-energy asymptotic [15,35]) in order to restore the whole function within the Padé approach [36,37,38] by properly extending the treatment elaborated more than a decade ago for the *massive three-loop polarization function* [39].

Within this method it is customary to deal with a "physically" normalized polarization operator $\hat{\Pi}$ defined such that

$$\hat{\Pi}(M, Q, a_s) = \Pi(M, Q, a_s) - \Pi(M, Q = 0, a_s),$$

where $\Pi(M, Q, a_s)$ is defined by eq.(3) with the use of the pole quark mass M mass instead of $\overline{\mathrm{MS}}$ renormalized quark mass m (see [40,41,42]). Using the results for $\Pi(M, Q = 0, a_s)$ as listed in [32,33] we arrive at the following asymptotic behavior of the (four-loop part of) $\hat{\Pi}(M, Q, a_s)$ at $Q \to \infty$.

$$\begin{aligned}\hat{\Pi}^0_3 \;=\;& \hat{L}^3\,(-0.0638534543 \\ &+0.007739812639345\, n_f - 0.0002345397769\, n_f^2) \\ &+\hat{L}^2\,(0.219061347 - 0.0264409511\, n_f \\ &\quad +0.000486744424\, n_f^2) \\ &+\hat{L}\,(-0.4620927910 + 0.1067886396\, n_f \\ &\quad 0.0021836455422\, n_f^2) + H_0^{(3)},\end{aligned} \tag{26}$$

$$\begin{aligned}H_0^{(3)} \;=\;& -11.4121461108 \\ &+1.4413529302\, n_f - 0.032814657849\, n_f^2,\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\hat{\Pi}^2_3}{Q^2} \;=\;& \frac{\hat{L}^3}{z}\,(-0.4511958959073 \\ &+0.03588458587\, n_f - 0.00070361933\, n_f^2) \\ &+\frac{\hat{L}^2}{z}\,(3.084755098861 \\ &-0.2601632475816\, n_f + 0.004573525651\, n_f^2) \\ &+\frac{\hat{L}}{z}\,(-6.6515904245 \\ &+0.78236916962\, n_f - 0.01465873606\, n_f^2) \\ &\quad +\frac{H_1^{(3)}}{z},\end{aligned} \tag{28}$$

$$\begin{aligned}H_1^{(3)} \;=\;& -8.16060818463927 \\ &+1.0812664904869\, n_f - 0.031095026978\, n_f^2.\end{aligned} \tag{29}$$

To be in agreement with the notations of [28] we have used in (26,28) $\hat{L} \equiv \ln(Q^2/M^2)$, $\quad z \equiv$

		$n_f = 4$	$n_f = 5$
$H_0^{(3)}$	Padé	-6.122 ± 0.054	-4.989 ± 0.053
$H_0^{(3)}$	exact	-6.17176892	-5.02574791
$H_1^{(3)}$	Padé	-3.885 ± 0.417	-3.180 ± 0.405
$H_1^{(3)}$	exact	-4.33306265	-3.53165141

Table 3
Comparison of the Padé method predictions for $H_0^{(3)}$ and $H_1^{(3)}$ with exact results.

$-\frac{Q^2}{4\,M^2}$ and set the renormalization scale $\mu = M$. In addition, we have separated in eqs. (26,28) $\hat{L}$-dependent pieces (known since long) from the new $\hat{L}$-independent ones.

The authors of [28] did not have at their disposal the $\hat{L}$-independent terms $H_0^{(3)}$ and $H_1^{(3)}$ and, thus, did not use them. Instead, they were able to *reconstruct* these terms for two particular values of $n_f = 4, 5$ from their final Padé approximants. Table 3 compares their (approximate) results with our exact ones. We observe the full agreement (within accuracy of the Padé approach) between our (exact) results and approximate ones obtained in [28].

5.2. Discussion

The full result for $R(s)$ is composed from two parts: the four-loop polarization operator (including its constant, that is $\ln(Q^2)$ independent terms) and the five-loop anomalous dimension γ^{VV}. At the level of separate Feynman diagrams the evaluation of *both* parts is reduced to the direct calculation of four-loop massless propagators [14]. On the other hand, the input data used in [28] came from three different sources:
(i) the threshold behavior of the polarization operator;
(ii) the $\ln(Q^2)$ dependent part of the high energy limit of the four-loop polarization operator which technically was obtained by a calculation of *three-loop* massless propagators only;
(iii) the first two physical moments of the four-loop polarization operator which technically were obtained by a calculation of massive *tadpoles* only.

Thus, we consider the full agreement demonstrated by Table 3 as a non-trivial and completely independent confirmation of the correctness of the results of [1,2,3].

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